Zigzag Persistence and Fractals

Tony Zeng

University of Washington

November 2024

Background

Definition

A d-dimensional **simplex** is the convex hull of $d + 1$ points.

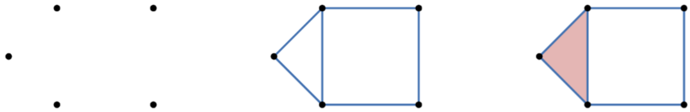
Background

Definition

A d-dimensional **simplex** is the convex hull of $d+1$ points.

Definition

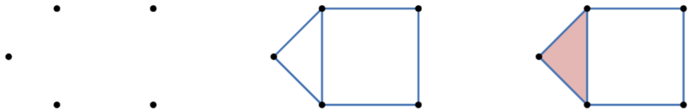
A **simplicial complex** is a set of simplices K closed under taking faces and intersections.



Background

Definition

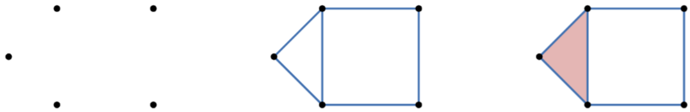
The **Vietoris-Rips Complex** (with parameter δ) of a set of points in a metric space is the simplicial complex consisting of all simplices of points whose diameter is less than δ.



Background

Definition

The **Vietoris-Rips Complex** (with parameter δ) of a set of points in a metric space is the simplicial complex consisting of all simplices of points whose diameter is less than δ.



Remark

Given $\delta_1 < \delta_2 < \ldots < \delta_m$, let K_i denote the Vietoris-Rips Complex associated with δ_i. Then $K_1 \leq K_2 \leq \ldots \leq K_m$ forms a filtration of simplicial complexes.

Persistent Homology

Definition

Given a filtration

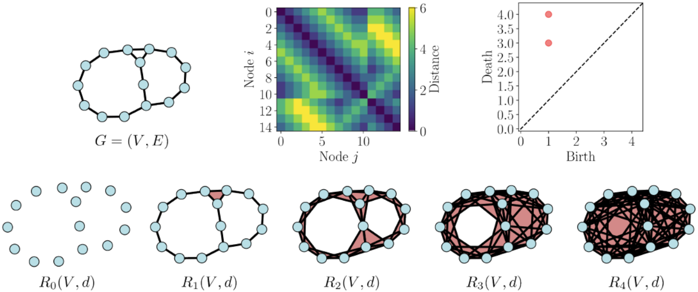
$$K_1 \leq K_2 \leq \cdots \leq K_m = K$$

of a simplicial complex K, the q-**dimensional persistent homology groups** are images of the maps

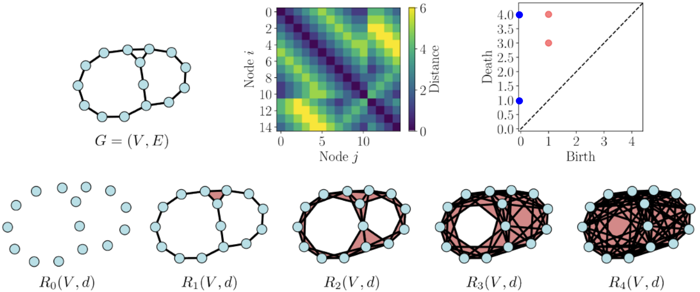
$$(\iota_{s,t})_* : H_q(K_s; \mathbb{F}) \to H_q(K_t; \mathbb{F}),$$

for $0 \leq s \leq t \leq m$. The ranks $\beta^q_{s,t} = \text{rank}(\iota_{s,t})_*$ are the **persistent Betti numbers**, which encode geometric/topological information.

Persistent Homology



Persistent Homology



Background

Definition

An **iterated function system** (or **IFS**) is a set of maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$f_i(x) = \rho_i U_i x + b_i,$$

where $\rho_i \in (-1, 1)$, U_i is orthogonal, and $b_i \in \mathbb{R}^n$.

Background

Definition

An **iterated function system** (or **IFS**) is a set of maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$f_i(x) = \rho_i U_i x + b_i,$$

where $\rho_i \in (-1, 1)$, U_i is orthogonal, and $b_i \in \mathbb{R}^n$.

We will be taking a look at a *fixed set* or *attractor* of a particular one-parameter family of IFSs,

Background

Definition

An **iterated function system** (or **IFS**) is a set of maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$f_i(x) = \rho_i U_i x + b_i,$$

where $\rho_i \in (-1, 1)$, U_i is orthogonal, and $b_i \in \mathbb{R}^n$.

We will be taking a look at a *fixed set* or *attractor* of a particular one-parameter family of IFSs, i.e. a set S such that

$$S = \bigcup_i f_i(S).$$

Iterated Function System

Theorem (Hutchinson, 1981)

For any IFS, there exists a unique non-empty compact fixed set S.

Iterated Function System

Theorem (Hutchinson, 1981)

For any IFS, there exists a unique non-empty compact fixed set S.

Consider the IFS given by

$$f_i(x) = \frac{1}{2}(R_{i\beta}x + e_1), \quad i \in \{-1, 0, 1\}$$

(where R_α denotes counterclockwise rotation about the origin by angle α).

Iterated Function System

Theorem (Hutchinson, 1981)

For any IFS, there exists a unique non-empty compact fixed set S.

Consider the IFS given by

$$f_i(x) = \frac{1}{2}(R_{i\beta}x + e_1), \quad i \in \{-1, 0, 1\}$$

(where R_α denotes counterclockwise rotation about the origin by angle α). What does the attractor of this IFS look like?

Sierpinski Family

Standard strategy to generate approximate images of the attractor of an IFS is a follows.

- Pick some initial point x_0.
- Let $x_{i+1} = f(x_i)$, where f is sampled uniformly at random from $\{f_i\}$.
- Plot $\{x_0, x_1, \ldots\}$.

Sierpinski Family

Standard strategy to generate approximate images of the attractor of an IFS is a follows.

- Pick some initial point x_0.
- Let $x_{i+1} = f(x_i)$, where f is sampled uniformly at random from $\{f_i\}$.
- Plot $\{x_0, x_1, \ldots\}$.

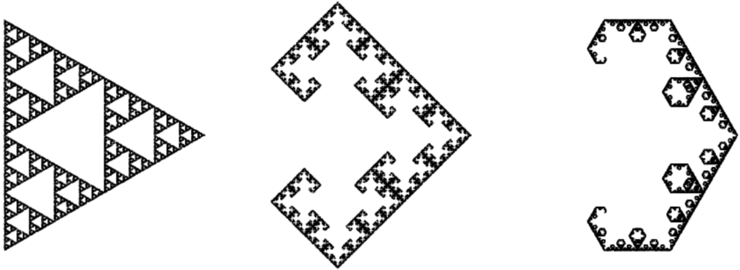
Example

We see that for $\beta = 2\pi/3$, the attractor of this IFS is the Sierpinski Triangle.

Sierpinski Family

What about other values of β?

What about other values of β?



What can we do?

Question

- What can we say about this family objects?
- For what parameters β are the attractors of this IFS topologically distinct?
- What can we detect?

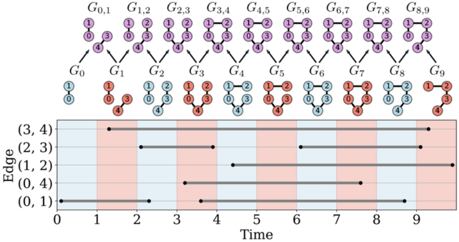
Zigzag Persistence

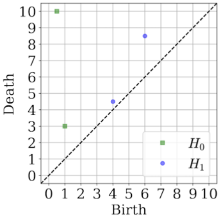
Myers, Muñoz, Khasawneh, and Munch[1] devised the notion of zigzag analysis to extract information from a time varying simplicial complex.

[1] Myers, Muñoz, Khasawneh, and Munch. (2023). Temporal network analysis using zigzag persistence. *EPJ Data Science*, 12(1):6

Zigzag Persistence

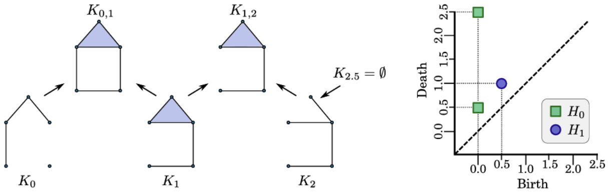
Myers, Muñoz, Khasawneh, and Munch[1] devised the notion of zigzag analysis to extract information from a time varying simplicial complex.





[1] Myers, Muñoz, Khasawneh, and Munch. (2023). Temporal network analysis using zigzag persistence. *EPJ Data Science*, 12(1):6

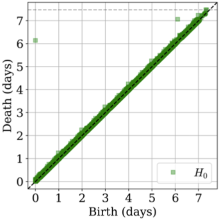
Myers, Muñoz, Khasawneh, and Munch[1] devised the notion of zigzag analysis to extract information from a time varying simplicial complex.



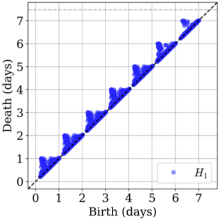
[1]Myers, Muñoz, Khasawneh, and Munch. (2023). Temporal network analysis using zigzag persistence. *EPJ Data Science*, 12(1):6

Zigzag Persistence


(a) Full Rail Travel Network.

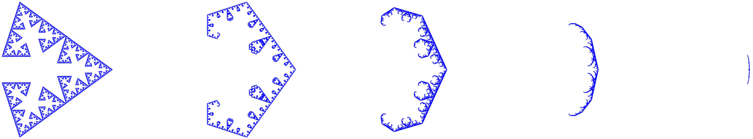


(b) Zero-dimensional zigzag persistence.

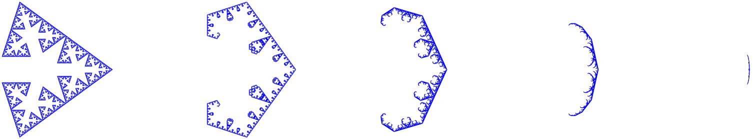


(c) One-dimensional zigzag persistence.

Simplicial Complex Construction

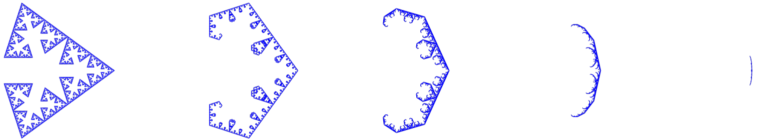


Simplicial Complex Construction



There are two things we need to proceed.

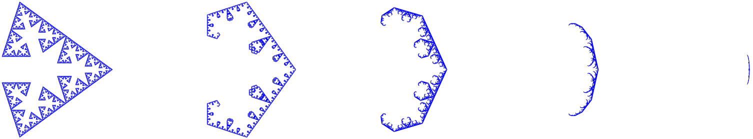
Simplicial Complex Construction



There are two things we need to proceed.

1. Reasonable coherent/consistent method of sampling from our attractors.

Simplicial Complex Construction



There are two things we need to proceed.

1. Reasonable coherent/consistent method of sampling from our attractors.
2. Sensible choice of distance for the Vietoris-Rips complex.

Sampling

Fix some initial point p and depth n. We take our sample to be

$$f_{i_1} \circ \cdots \circ f_{i_n}(p),$$

for all $(i_1, \ldots, i_n) \in \{-1, 0, 1\}^n$.

Sampling

Fix some initial point p and depth n. We take our sample to be

$$f_{i_1} \circ \cdots \circ f_{i_n}(p),$$

for all $(i_1, \ldots, i_n) \in \{-1, 0, 1\}^n$. This gives a reasonable sampling scheme due to continuity with respect to β.

Distance

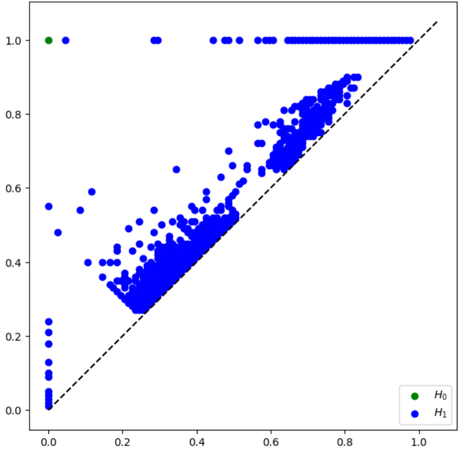
Note that our attractors are scaled differently depending on β. Therefore, the distance parameter for the Vietoris-Rips complex should also depend on β.

Distance

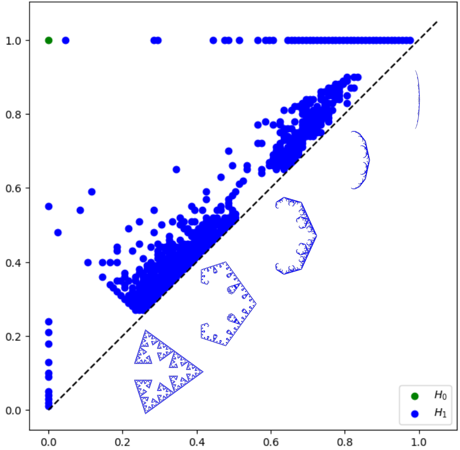
Note that our attractors are scaled differently depending on β. Therefore, the distance parameter for the Vietoris-Rips complex should also depend on β. Consider

$$\delta = \frac{2+\varepsilon}{2^n} \cdot \sin\left(\frac{\beta}{2}\right).$$

Zigzag Persistence Diagram



Zigzag Persistence Diagram



Thanks!